



Existence of two positive solutions of a singular nonlinear periodic boundary value problem

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Abstract

We consider the singular periodic boundary value problem for the second order scalar differential equation $x'' = f(t, x, x')$. Under the assumption that f has the repulsive singularity at $x = 0^+$, we present sufficient conditions for the existence of at least two different positive solutions of the problem. We use the lower and upper solutions method here. © 2000 Elsevier Science B.V. All rights reserved.

1. Introduction

This paper deals with the singular boundary value problem

$$x'' = f(t, x, x'), \quad (1)$$

$$x(0) = x(T), \quad x'(0) = x'(T), \quad (2)$$

where $f \in \text{Car}(J \times (\mathbb{R}^+ \times \mathbb{R}))$, $J = [0, T] \subset \mathbb{R}$, $\mathbb{R}^+ = (0, \infty)$, and f has a repulsive singularity at $x = 0^+$, i.e.

$$\lim_{x \rightarrow 0^+} f(t, x, y) = +\infty \quad \text{for a.e. } t \in J \text{ and each } y \in \mathbb{R}.$$

We are interested in positive solutions of (1), (2), because we have been motivated by a problem from the Theory of Nonlinear Elasticity modelling radial oscillations of an elastic spherical membrane made up of a Neo-Hookean material, and subjected to an internal pressure. The oscillations are governed by the scalar equation

$$x'' = p(t)x^2 - x + x^{-5}, \quad (3)$$

where the pressure $p : \mathbb{R} \rightarrow (0, \infty)$ is continuous and T -periodic and $x(t)$ is the ratio of the deformed radius of the membrane at time t with respect to the undeformed radius, and hence $x(t) > 0$ on \mathbb{R} .

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See [3] or [1]. Here, we prove the existence of at least two different positive solutions to the more general problem (1), (2) using theorems of Rachunkova [2] which are based on a connection between the lower and upper solutions and the topological degree of an operator associated to (1), (2). We generalize results of Gaete and Manasevich [1], where the equation

$$x'' = g(t, x), \quad (4)$$

with $g : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ continuous and T -periodic in t , is considered and where the authors prove the existence of two positive periodic solutions for (4) by a variational method. Such an approach cannot be used for (1), (2).

2. Notation and definitions

For $k \in \mathbb{N} \cup \{0\}$, $C^k(J)$ is the Banach space of functions having continuous k th derivatives on J with the norm

$$\|x\|_{C^k} = \sum_{i=0}^k \max\{|x^{(i)}(t)| : t \in J\} \quad \text{for } x \in C^k(J).$$

$L_1(J)$ is the Banach space of Lebesgue integrable functions on J with the norm

$$\|x\|_{L_1} = \int_0^T |x(t)| dt \quad \text{for } x \in L_1(J).$$

By $AC^k(J)$ we denote the set of functions having absolutely continuous k th derivatives on J , and we use $C(J)$ or $AC(J)$ instead of $C^0(J)$ or $AC^0(J)$. For $B \subset \mathbb{R}$, the symbol $\text{Car}(J \times (B \times \mathbb{R}))$ denotes the set of functions satisfying the Carathéodory conditions on $J \times (B \times \mathbb{R})$, i.e. $f(\cdot, x, y) : J \rightarrow \mathbb{R}$ is measurable for all $x \in B, y \in \mathbb{R}$, $f(t, \cdot, \cdot) : B \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous for a.e. $t \in J$ and $\sup\{|f(t, x, y)| : (x, y) \in K\} \in L_1(J)$ for any compact set $K \subset (B \times \mathbb{R})$. For $\varphi \in L_1(J)$ we denote by $\bar{\varphi}$ the mean value $(1/T) \int_0^T \varphi(s) ds$.

By a solution of (1), (2) we understand a function $u \in AC^1(J)$ satisfying (1) for a.e. $t \in J$, as well as the boundary conditions (2). By a lower solution of (1), (2) we mean a function $\sigma_1 \in AC^1(J)$ satisfying

$$\sigma_1''(t) \geq f(t, \sigma_1(t), \sigma_1'(t)) \quad \text{for a.e. } t \in J \quad (5)$$

and the boundary conditions

$$\sigma_1(0) = \sigma_1(T), \quad \sigma_1'(0) \geq \sigma_1'(T). \quad (6)$$

Similarly, by an upper solution of (1), (2) we mean a function $\sigma_2 \in AC^1(J)$ satisfying

$$\sigma_2''(t) \leq f(t, \sigma_2(t), \sigma_2'(t)) \quad \text{for a.e. } t \in J \quad (7)$$

and the boundary conditions

$$\sigma_2(0) = \sigma_2(T), \quad \sigma_2'(0) \leq \sigma_2'(T). \quad (8)$$

3. Assumptions

We will use the following assumptions in our lemmas and theorems:

$$f \in \text{Car}(J \times (\mathbb{R}^+ \times \mathbb{R})), \quad (9)$$

$$\exists m \in L_1(J): f(t, x, y) > m(t) \quad \text{for a.e. } t \in J, \forall (x, y) \in \mathbb{R}^+ \times \mathbb{R}, \quad (10)$$

$$\exists g \in C(\mathbb{R}^+): f(t, x, y) > g(x) \quad \text{for a.e. } t \in J, \forall (x, y) \in \mathbb{R}^+ \times [0, \infty), \quad (11)$$

where $g(x) > 0$ for all $x \in (0, r)$ and for some $r \in \mathbb{R}^+$;

$$G(0^+) = -\infty, G(\infty) = -\infty \quad \text{where } G(x) = \int_r^x g(s) ds \quad \text{for } x \in \mathbb{R}^+, \quad (12)$$

$$\exists \sigma \in (r, \infty): f(t, \sigma, 0) \leq 0 \quad \text{for a.e. } t \in J, \quad (13)$$

$$\exists \varepsilon \in \mathbb{R}^+: f(t, \sigma + \varepsilon, 0) \leq 0 \quad \text{for a.e. } t \in J, \quad (14)$$

$$\exists \varphi \in L_1(J), \exists A \in \mathbb{R}^+: \bar{\varphi} \geq 0, f(t, x, y) \geq \varphi(t) \quad \text{for a.e. } t \in J, \forall x > A, \forall y \in [-\|m\|_{L_1}, \|m\|_{L_1}]. \quad (15)$$

Suppose, that the function p on the right-hand side of (3) satisfies

$$0 < p_1 = \min_{t \in J} p(t) < \max_{t \in J} p(t) = p_2 < 6/7^{7/6} \quad (16)$$

and put

$$b(t, x) = p(t)x^2 - x + x^{-5}.$$

Then (16) guarantees that b has both signs which is necessary for the existence of periodic solutions to (3).

Lemma 3.1. *Under assumption (16) $f = b$ fulfils all conditions (9)–(15).*

Proof. Condition (9) follows from the fact that $b \in C(J \times \mathbb{R}^+)$. Condition (10): $b(t, x) > p_1 x^2 - x \geq -1/4 p_1$ on $J \times \mathbb{R}^+$. Condition (11): $b(t, x) > -x + x^{-5} = g(x)$ on $J \times \mathbb{R}^+$, where $g \in C(\mathbb{R}^+)$ and $g(x) > 0$ for all $x \in (0, 1)$. Condition (12): $G(x) = \int_1^x g(s) ds = -x^{-4}/4 - x^2/2 + 3/4$ and $G(0^+) = -\infty, G(\infty) = -\infty$. Conditions (13) and (14): $b(t, x) \leq p_2 x^2 - x + x^{-5} = \psi_{p_2}(x)$ on $J \times \mathbb{R}^+$. For $p_2 = 6/7^{7/6}$ the function ψ_{p_2} is nonnegative on \mathbb{R}^+ and it has just one minimal zero value at $x = 7^{1/6}$. For $p_2 \in (0, 6/7^{7/6})$ we can find an interval $(\alpha, \beta) \subset (1, \infty)$ such that $\psi_{p_2}(x) < 0$ for all $x \in (\alpha, \beta)$. Thus we can choose numbers $\sigma, \sigma + \varepsilon \in (\alpha, \beta)$ satisfying (13) and (14). Condition (15): $b(t, x) > p_1 x^2 - x > 2/p_1$ for all $t \in J, x > A$, where $A = 2/p_1$. \square

4. A priori estimates

Lemma 4.1. *Suppose (9) and (10) hold. Then for any positive solution u of the problem (1), (2) the estimate*

$$\|u'\|_C < \|m\|_{L_1} \quad (17)$$

is true.

Proof. Let u be a positive solution of (1), (2). Then there exists $t_0 \in J$ such that $u'(t_0) = 0$. Eq. (10) implies that $m(t) < u''(t)$ for a.e. $t \in J$. Therefore

$$-\int_{t_0}^T |m(t)| dt < u'(t) \quad \text{for any } t \in [t_0, T] \text{ and for } t = 0.$$

Thus $-\|m\|_{L_1} < u'(t)$ on J . Similarly

$$u'(t) < \int_0^{t_0} |m(t)| dt \quad \text{for any } t \in [0, t_0] \text{ and for } t = T.$$

So, $u'(t) < \|m\|_{L_1}$ on J , and (17) is proved. \square

Lemma 4.2. Suppose (9)–(12) hold. Then there exists $c^* \in \mathbb{R}^+$ such that, for any positive solution u of (1), (2),

$$0 < c^* < u(t) \quad \text{for any } t \in J. \quad (18)$$

Proof. Let u be a positive solution of (1), (2). According to (2) we can extend u periodically on \mathbb{R} . Suppose that there exists $s \in J$ such that

$$0 < u(s) \leq r. \quad (19)$$

Using Lemma 4.1 we get (17) which together with (19) gives the estimate

$$\|u\|_C < \|m\|_{L_1} T + r = r^*. \quad (20)$$

Suppose that

$$\min_{t \in J} u(t) = u(t_0) \in (0, r). \quad (21)$$

Then $u'(t_0) = 0$. Let $\varepsilon > 0$ be such that $u'(t) = 0$ for all $t \in I_\varepsilon = [t_0, t_0 + \varepsilon]$. Then $u(t) = u(t_0)$ and $u''(t) = 0$ for all $t \in I_\varepsilon$. So, by (11), we get

$$0 = \int_{I_\varepsilon} f(t, u(t_0), 0) dt > \int_{I_\varepsilon} g(u(t_0)) dt > 0,$$

a contradiction. Suppose that we can find a sequence $\{t_n\} \subset \mathbb{R}$ such that $t_n \rightarrow t_{0+}$ and $u'(t_n) = 0$ for all $n \in \mathbb{N}$. Then there exists $m \in \mathbb{N}$ such that $u(t) \in (0, r)$ and $u'(t) > 0$ for all $t \in (t_m, t_{m+1})$. Therefore

$$0 = \int_{t_m}^{t_{m+1}} u''(t) dt = \int_{t_m}^{t_{m+1}} f(t, u(t), u'(t)) dt > \int_{t_m}^{t_{m+1}} g(u(t)) dt > 0,$$

a contradiction. So, we have proved that u' is positive on some right neighbourhood of t_0 . Let us show that there exists t_1 such that

$$0 < t_1 - t_0 < T, \quad u'(t_1) = 0 \quad \text{and} \quad u'(t) > 0 \quad \text{for all } t \in (t_0, t_1). \quad (22)$$

If $u'(t) < 0$ on $[0, t_0)$ and $u'(t) > 0$ on $(t_0, T]$, then this contradicts (2). Hence $t_1 \in (t_0, T]$ or $u'(\tau) = 0$ for some $\tau \in [0, t_0)$ and we set $t_1 = \tau + T$.

Now, using (11) and (22), we have

$$u'' \cdot u' = f(t, u(t), u'(t)) \cdot u'(t) > g(u(t)) \cdot u'(t) \quad \text{for a.e. } t \in (t_0, t_1)$$

and

$$\int_{t_0}^{t_1} u'(t)u''(t) dt > \int_{t_0}^{t_1} g(u(t))u'(t) dt.$$

Thus $0 = \frac{1}{2}(u'^2(t_1) - u'^2(t_0)) > G(u(t_1)) - G(u(t_0))$, i.e.

$$G(u(t_0)) > G(u(t_1)). \quad (23)$$

We can suppose, without loss of generality, that $g(x) < 0$ for all $x > r$. Then G is increasing on $(0, r)$ and decreasing on (r, ∞) , which together with (21) and (23) imply that $u(t_1) > r$. In view of (20) $G(r^*) < G(u(t_1))$ and there exists just one number $c^* \in (0, r)$ such that

$$G(c^*) = G(r^*). \quad (24)$$

Therefore $G(c^*) < G(u(t_0))$, which gives (18). We have proved (18) for any positive solution satisfying (19). If some positive solution v of (1), (2) does not have property (19), then $\min_{t \in J} v(t) > r$, and so we get (18) immediately. \square

Lemma 4.3. Suppose (9)–(12) hold and set $r^* \geq \|m\|_{L_1} T + r$, $c^* = G^{-1}(G(r^*)) \in (0, r)$. Further put

$$f^*(t, x, y) = \begin{cases} f(t, x, y) & \text{if } x > c^*, \\ f(t, c^*, y) + c^* - x & \text{if } x \leq c^*, \end{cases} \quad (25)$$

for a.e. $t \in J$ and all $x, y \in \mathbb{R}$ and consider the equation

$$u'' = f^*(t, u, u'). \quad (26)$$

Then any solution u of problem (26), (2) is a solution of (1), (2) and satisfies estimate (18).

Proof. $f^* \in \text{Car}(J \times \mathbb{R}^2)$ and f^* satisfies (10) for all $x \in \mathbb{R}$. Therefore, the estimate (17) is true for any solution u of (26), (2). The proof of this assertion is similar to that of Lemma 4.1. Let u be a solution of (26), (2) and suppose that $\min_{t \in J} u(t) = u(t_0) < c^*$. Then we can argue like in the proof of Lemma 4.2 and get a point t_1 satisfying (22). If $u(t_1) \leq c^*$, then

$$0 = \int_{t_0}^{t_1} u'(s) ds = \int_{t_0}^{t_1} (f(s, c^*, u'(s)) + c^* - u(s)) ds > \int_{t_0}^{t_1} g(c^*) ds > 0,$$

a contradiction. Thus, suppose that $u(t_1) > c^*$. Then there exists $t^* \in (t_0, t_1)$ such that $u(t^*) = c^*$ and

$$\begin{aligned} 0 = \int_{t_0}^{t_1} u''(s)u'(s) ds &= \int_{t_0}^{t^*} (f(s, c^*, u'(s)) + c^* - u(s))u'(s) ds \\ &\quad + \int_{t^*}^{t_1} f(s, u(s), u'(s))u'(s) ds > \int_{t^*}^{t_1} g(u(s))u'(s) ds = G(u(t_1)) - G(c^*). \end{aligned}$$

Analogically, like in the proof of Lemma 4.2 we deduce that $u(t_1) \in (r, r^*)$ and $G(r^*) < G(u(t_1))$. Since $G(r^*) = G(c^*) > G(u(t_1))$, we get a contradiction. Thus u satisfies (18) and it is a solution of (1), (2), as well. \square

5. Main results

First, we will study a regular problem for the equation

$$u'' = h(t, u, u'), \quad (27)$$

with $h \in \text{Car}(J \times \mathbb{R}^2)$.

Theorem 5.1. *Let σ_1 and σ_2 be lower and upper solutions of (27), (2). Further suppose that for a.e. $t \in J$ and all $x, y \in \mathbb{R}$*

$$h(t, x, y) \geq -\rho(t, |x| + |y|), \quad (28)$$

where $\rho \in \text{Car}(J \times \mathbb{R}^+)$ is a nonnegative function, which is nondecreasing and sublinear in its second variable, i.e.

$$\lim_{z \rightarrow \infty} \frac{1}{z} \int_0^T \rho(t, z) dt = 0. \quad (29)$$

I. Then problem (27), (2) has at least one solution u .

II. (a) If $\sigma_1(t) \leq \sigma_2(t)$ for all $t \in J$, then $\sigma_1(t) \leq u(t) \leq \sigma_2(t)$ for all $t \in J$.

(b) If $\sigma_2(t) \leq \sigma_1(t)$ for all $t \in J$, then there exists $t_u \in J$ such that $\sigma_2(t_u) \leq u(t_u) \leq \sigma_1(t_u)$.

(c) If σ_1 and σ_2 are not ordered on J , then there exist $t_u, s_u \in J$ such that $\sigma_2(t_u) \leq u(t_u)$ and $u(s_u) \leq \sigma_1(s_u)$.

Proof. Let us set $H(\mu) = (1/\mu) \int_0^T \rho(s, \mu(T+1) + r) ds$ for $\mu \in (0, \infty)$ and $r = \|\sigma_1\|_C + \|\sigma_2\|_C$. Condition (29) implies that there exists $\mu^* \in (0, \infty)$ with

$$H(\mu) < 1 \quad \text{for all } \mu \geq \mu^*. \quad (30)$$

Now, let us consider the auxiliary differential equation

$$u'' = h^*(t, u, u'), \quad (31)$$

where

$$h^*(t, x, y) = \chi(|x| + |y|, r^*) h(t, x, y) \quad \text{for a.e. } t \in J \text{ and all } x, y \in \mathbb{R},$$

$$r^* = \mu^*(T+1) + r,$$

$$\chi(s, \phi) = \begin{cases} 1 & \text{for } 0 \leq s \leq \phi, \\ 2 - s/\phi & \text{for } \phi < s < 2\phi, \\ 0 & \text{for } s \geq 2\phi. \end{cases}$$

Since there exists $m^* \in L_1(J)$ such that $|h^*(t, x, y)| < m^*(t)$ for a.e. $t \in J$ and all $x, y \in \mathbb{R}$, we can apply Theorem 6 from Rachunkova [2] onto problem (31), (2) and get a solution of this problem which fulfills the assertion II of our Theorem 5.1. Let us prove that u is a solution of (27). Putting $\mu = \|u'\|_C$, we get $\|u\|_C \leq \mu T + r$. In view of (2) we can find $t_0 \in J$ such that $u'(t_0) = 0$ and (28) gives

$$u''(t) \geq -\rho(t, |u| + |u'|) \quad \text{for a.e. } t \in J. \quad (32)$$

In a way similar to the proof of Theorem 8 in [2] we get from (32) by integration

$$|u'(t)| \leq \int_0^T \rho(t, \mu(T+1) + r) dt \quad \text{for all } t \in J,$$

and so $1 \leq H(\mu)$. The latter inequality together with (30) imply that $\mu < \mu^*$. Thus $\|u\|_C + \|u'\|_C < r^*$ and u satisfies (27). \square

Theorem 5.2. *Suppose (9)–(12) hold and let a positive function σ_1 be a lower solution of problem (1), (2). Then this problem has at least one positive solution.*

Proof. Let r^* and c^* be the numbers from Lemma 4.3. Without loss of generality, we can suppose $c^* \leq \min_{t \in J} \sigma_1(t)$. (Otherwise we take instead of r^* and c^* numbers r^{**} and c^{**} , which satisfy $c^{**} = \min_{t \in J} \sigma_1(t) < c^*$ and $r^{**} = G^{-1}(G(c^{**}) \in (r^*, \infty))$. Now, let us consider the auxiliary problem (26), (2). By (9), (10) and (25), $f^* \in \text{Car}(J \times \mathbb{R}^2)$ and f^* satisfies (28) with $\rho(t, z) = |m(t)|$. Since c^* and σ_1 are upper and lower solution of (26), (2), respectively, Theorem 5.1 implies that (26), (2) has a solution u with

$$c^* \leq u(t_u) \leq \sigma_1(t_u) \quad \text{for some } t_u \in J. \quad (33)$$

By Lemma 4.3, u is a solution of (1), (2) and satisfies (18). \square

Theorem 5.3. *Suppose (9), (10), (15) hold and let a positive function σ_1 be a lower solution of (1), (2). Then this problem has at least one solution u with $\sigma_1(t) \leq u(t)$ for all $t \in J$.*

Proof. Let c^* be a positive number satisfying $c^* \leq \min_{t \in J} \sigma_1(t)$ and let f^* be given by (25). Let us consider problem (26), (2). Without loss of generality, we can suppose that A in (15) satisfies $A > \max_{t \in J} \sigma_1(t)$ and we can check that

$$\sigma_2(t) = A + 2T\|\varphi\|_{L_1} - t \int_0^T \int_0^\tau \varphi(s) ds d\tau + \int_0^t \int_0^\tau \varphi(s) ds d\tau$$

is an upper solution of (26), (2). So, by Theorem 5.1, problem (26), (2) has a solution u lying between σ_1 and σ_2 on J . In view of (25), u is a solution of (1), as well. \square

Theorem 5.4. *Suppose (9)–(12), (15) hold and let positive functions σ_1 and $\sigma_1 + \varepsilon$, where $\varepsilon \in \mathbb{R}^+$, be lower solutions of the problem (1), (2). Then this problem has at least two positive solutions.*

Proof. Theorem 2 implies the existence of a positive solution u of (1), (2) which satisfies (33). Theorem 5.3 gives the existence of a solution v of (1), (2) which has the property

$$\sigma_1(t) + \varepsilon \leq v(t) \quad \text{for all } t \in J. \quad (34)$$

According to (33) and (34) we see that u and v are different solutions. \square

Theorem 5.5. *Suppose (9)–(15) hold. Then problem (1), (2) has at least two positive solutions.*

Proof. Since (13) and (14) imply that the numbers σ and $\sigma + \varepsilon$ are constant positive lower solutions of (1), (2), the assertion follows from Theorem 5.4. \square

Corollary 5.6. *Suppose (16) holds. Then Eq. (4) has at least two positive T -periodic solutions.*

Proof. This assertion is a direct consequence of Theorem 5.4 and Lemma 3.1. \square

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